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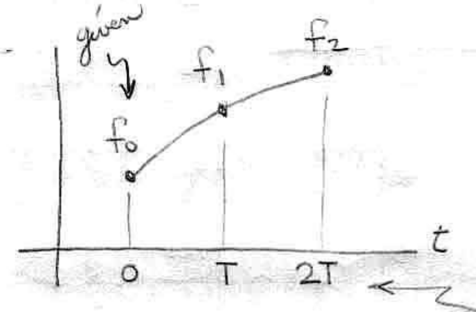
# Piece Wise Quadratic Time Domain Analysis



these are arbitrary, we could have used -T, 0, and T

Starting

$f_0$ = initial condition

$f_1$ = unknown future value

$f_2$ = unknown future value

$$f(t) = at^2 + bt + C$$

$$\boxed{f(0) = f_0 = C}$$

$$f(T) = f_1 = aT^2 + bT + C$$

and

$$f(2T) = f_2 = 4aT^2 + 2bT + C$$

then we have

$$-2f_1 + 2f_0 = -2aT^2 - \cancel{2bT} \nearrow 0$$

$$f_2 - f_0 = 4aT^2 + \cancel{2b}T$$

summing these

or

$$f_2 - 2f_1 + f_0 = 2aT^2$$

$$\boxed{a = \frac{1}{2T^2}[f_0 - 2f_1 + f_2]}$$

$$4f_1 - 4f_0 = \cancel{4aT^2} + 4bT \nearrow 0$$

$$-f_2 + f_0 = -\cancel{4aT^2} - 2bT$$

summing these

$$-3f_0 + 4f_1 - f_2 = 2bT$$

$$\boxed{b = \frac{1}{2T}[-3f_0 + 4f_1 - f_2]}$$

Note that interpolation can be performed using:

$$f(t) = \frac{t^2}{2T^2}(f_0 - 2f_1 + f_2) + \frac{t}{2T}(-3f_0 + 4f_1 - f_2) + f_0$$

$$f'(t) = \frac{t}{T^2}(f_0 - 2f_1 + f_2) + \frac{1}{2T}(-3f_0 + 4f_1 - f_2)$$

$$\int_{t_1}^{t_2} f(t)dt = \frac{t_1^3 - t_2^3}{6T^2}(f_0 - 2f_1 + f_2) + \frac{t_1^2 - t_2^2}{4T}(-3f_0 + 4f_1 - f_2) + (t_1 - t_2)f_0$$

$t, t_1, t_2, T$ are in seconds for MKS

We shall solve for $f_1$ and $f_2$ simultaneously. This will require both differentiation and integration at $t = T$ and $t = 2T$

$$f'(t) = 2at + b$$

$$f'(T) = \frac{1}{T2}\left[f_0 - 2f_1 + f_2\right]T + \frac{1}{2T}\left[-3f_0 + 4f_1 - f_2\right]$$

$$f'(T) = \frac{1}{T}\left[f_0\left(1 - \frac{3}{2}\right) + f_1\left(-2 + 2\right) + f_2\left(1 - \frac{1}{2}\right)\right]$$

$$\boxed{f'(T) = \frac{1}{2T}\left[-f_0 + f_2\right]}$$

$$f'(2T) = \frac{1}{T^2}\left[f_0 - 2f_1 + f_2\right]2T + \frac{1}{2T}\left[-3f_0 + 4f_1 - f_2\right]$$

$$f'(2T) = \frac{1}{T}\left[f_0\left(2 - \frac{3}{2}\right) + f_1\left(-4 + 2\right) + f_2\left(2 - \frac{1}{2}\right)\right]$$

$$\boxed{f'(2T) = \frac{1}{2T}\left[f_0 - 4f_1 + 3f_2\right]}$$

$$\int_0^T f(t)dt = \int_0^T (at^2 + bt + c)\,dt = \frac{1}{3}aT^3 + \frac{1}{2}bT^2 + cT + \text{cons.}$$

$$= \frac{T^3}{3}\frac{1}{2T^2}\left[f_0 - 2f_1 + f_2\right] + \frac{T^2}{2}\cdot\frac{1}{2T}\left[-3f_0 + 4f_1 - f_2\right] + Tf_0 + \text{constant}$$

$$= T\left[f_0\left(\frac{1}{6} - \frac{3}{4} + 1\right) + f_1\left(-\frac{2}{6} + \frac{4}{4}\right) + f_2\left(\frac{1}{6} - \frac{1}{4}\right)\right] + \text{const.}$$

$$\boxed{\int_0^T f(t)dt = T\left[\frac{5}{12}f_0 + \frac{8}{12}f_1 - \frac{1}{12}f_2\right] + \text{const.}}$$

likewise

$$\boxed{\int_0^{2T} f(t)dt = T\left[\frac{1}{3}f_0 + \frac{4}{3}f_1 + \frac{1}{3}f_2\right] + \text{const.}}$$

We have defined all the time equations needed to write a piecewise solution for a network stepping through time.

Nodal $\Sigma i's = 0$ Equations:

Resistor: $i_1 = \frac{1}{R} v_1 \quad i_2 = \frac{1}{R} v_2$ (no time coupling)

Capacitor: $i_1 = \frac{C}{T}\left[-\frac{v_0}{2} + \frac{v_2}{2}\right] \quad i_2 = \frac{C}{T}\left[\frac{v_0}{2} - 2v_1 + \frac{3}{2}v_2\right]$

Inductor: $i_1 = i_0 + \frac{T}{L}\left[\frac{5}{12}v_0 + \frac{8}{12}v_1 - \frac{1}{12}v_2\right]$

$$i_2 = i_0 + \frac{T}{L}\left[\frac{1}{3}v_0 + \frac{4}{3}v_1 + \frac{1}{3}v_2\right]$$

Loop $\Sigma v's = 0$ Equations:

Resistor: $v_1 = R i_1 \quad v_2 = R i_2$ (no time coupling)

Capacitor: $v_1 = v_0 + \frac{T}{C}\left[\frac{5}{12}i_0 + \frac{8}{12}i_1 - \frac{1}{12}i_2\right]$

$$v_2 = v_0 + \frac{T}{C}\left[\frac{1}{3}i_0 + \frac{4}{3}i_1 + \frac{1}{3}i_2\right]$$

Inductor: $v_1 = \frac{L}{T}\left[-\frac{i_0}{2} + \frac{i_2}{2}\right] \quad v_2 = \frac{L}{T}\left[\frac{i_0}{2} - 2i_1 + \frac{3}{2}i_2\right]$

where

| Time | current | voltage | |
|---|---|---|---|
| 0 | $i_0$ | $v_0$ | ← initial conditions |
| T | $i_1$ | $v_1$ | ← time T |
| 2T | $i_2$ | $v_2$ | ← time 2T |

after solution, let $i_0 = i_2$ and $v_0 = v_2$ and repeat the process.

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Example 1.

Matrix for a parallel RLC in nodal form.
(from nodal equations on p. 3)

$$\underbrace{\begin{bmatrix} \left(\frac{v_0}{2}\right)\frac{C}{T} - i_0 - \frac{5}{12} v_0 \frac{T}{L} \\ \left(-\frac{v_0}{2}\right)\frac{C}{T} - i_0 - \frac{1}{3} v_0 \frac{T}{L} \end{bmatrix}}_{\text{injection current and initial condx.}} = \begin{bmatrix} \frac{1}{R} + \frac{8}{12}\frac{T}{L} & \frac{1}{2}\frac{C}{T} - \frac{1}{12}\frac{T}{L} \\ -2\frac{C}{T} + \frac{4}{3}\frac{T}{L} & \frac{1}{R} + \frac{3}{2}\frac{C}{T} + \frac{1}{3}\frac{T}{L} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \begin{matrix} \leftarrow \text{KCL at time } T \\ \leftarrow \text{KCL at time } 2T \end{matrix}$$

$R \to \infty$

if $v_0 = 10$, and 100 step with $2T = .01$; the $v_2 = 9.99983637$ after 1 cycle.

solving only for $v_2$; assuming both $v_0$ and $v_1$ are known we have

$$\underbrace{\left[ -\frac{v_0}{2}\frac{C}{T} + 2v_1\frac{C}{T} - i_0 + \frac{1}{12} v_0 \frac{T}{L} - \frac{8}{12} v_1 \frac{T}{L} \right]}_{\text{initial conditions}} = \left[ \overset{R\to\infty}{\frac{1}{R}} + \frac{3}{2}\frac{C}{T} + \frac{5}{12}\frac{T}{L} \right] \left[ v_2 \right] \leftarrow \begin{matrix} \text{KCL at time} \\ T \text{ to } 2T \end{matrix}$$

this formulation of 200 steps with $T = .005$; $v_2 = 9.9997867$ after 1 cycle

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## Example 2

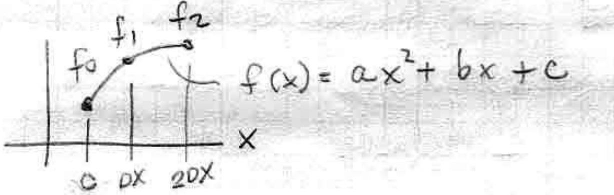
Solve $y'' = \frac{1}{2}(X + y + y' + 2)$ $y(0) = 0$ $y'(0) = 0$

the exact solution to this is: $y = e^{X} - X - 1$

for PQ let $f(x) = y'(x) \rightarrow y(x) = \int f(x)dx$

then $f'(X) = \frac{1}{2}\left(X + \int f(x)dx + \underset{\nwarrow \text{ constant of integration}}{Y} + f(x) + 2\right)$ ①

we solve for <u>two</u> values of $f(x)$ simultaneously



at X = DX we can expand ① to this equation:

$$\underbrace{\frac{1}{DX}\left(-\frac{f_0}{\cancel{2}} + \frac{f_2}{\cancel{2}}\right)}_{f'(DX)} = \frac{1}{\cancel{2}}\left[(X+DX) + \underbrace{DX\left(\frac{5}{12}f_0 + \frac{8}{12}f_1 - \frac{1}{12}f_2\right) + Y}_{\int_0^{DX} f(x)dx + Y} + f_1 + 2\right)\right]$$

at X = 2DX we can expand ① to this equation:

$$\underbrace{\frac{1}{DX}\left(\frac{f_0}{\cancel{2}} - \overset{4}{\cancel{2}}f_1 + \frac{3}{\cancel{2}}f_2\right)}_{f'(2DX)} = \frac{1}{\cancel{2}}\left[(X+2DX) + \underbrace{DX\left(\frac{1}{3}f_0 + \frac{4}{3}f_1 + \frac{1}{3}f_2\right) + Y}_{\int_0^{2DX} f(x)\,dx + Y} + f_2 + 2)\right]$$

Then in matrix form

$$\underbrace{\begin{bmatrix} -\frac{f_0}{DX} - X - DX - \frac{5}{12}DXf_0 - Y - 2 \\ \frac{f_0}{DX} - X - 2DX - \frac{1}{3}DXf_0 - Y - 2 \end{bmatrix}}_{\text{boundary condx.}} = \underbrace{\begin{bmatrix} \frac{8}{12}DX + 1 & -\frac{1}{DX} - \frac{1}{12}DX \\ \frac{4}{DX} + \frac{4}{3}DX & \frac{-3}{DX} + \frac{1}{3}DX + 1 \end{bmatrix}}_{\text{matrix coeff.}} \underbrace{\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}}_{\text{variables}}$$

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after each iteration let:

$$Y = Y + DX\left(\frac{1}{3}f_0 + \frac{4}{3}f_1 + \frac{1}{3}f_2\right)$$

then let

and $f_0 = f_2$

$$X = X + 2DX$$

Results of simulation

| | | | — Absolute Error — | |
|---|---|---|---|---|
| X | "Exact"<br>$Y = e^x - x - 1$ | Absolute Error Runge-Kutta * | PQ<br>100 Steps (to 1.0) | PQ<br>10 steps (to 1.0) |
| .2 | .021 4028 | .000 0020 | .000 0000 | .000 0144 |
| .4 | .091 8247 | .000 0043 | .000 0004 | .000 0534 |
| .6 | .222 1188 | .000 0071 | .000 0010 | .000 1256 |
| .8 | .425 5409 | .000 0106 | .000 0021 | .000 2422 |
| 1.0 | .718 2818 | .000 0150 | .000 0037 | .000 4177 |

* Erwin Kreyszig, Advanced Engineering Mathematics, Third Edition, page 671, Table 15.

Although PQ requires more steps to achieve accuracy, this is easily controlled by the user.

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The following are advantages of PΦ:

1) Simple formulation into a direct matrix solution
2) No special "starting" process is needed
3) Variable time steps can be easily incorporated
4) Cumulative error is small, known, and controllable
5) Interpolation can be used to set boundary values immediately before a switching event
6) "Instant" circuit changes are easily modeled
7) Nonlinear circuit elements can be modeled
8) Spectral analysis can be done with FFT directly from the PΦ solution.
9) Higher order differential problems can be modeled using more than two forward times.
10) PΦ can be applied to <u>any</u> differential equation problem, linear or nonlinear!
11) Random processes such as noise are easily introduced into the problem. Statistics are collected by letting the problem run for a while.
12) PΦ will free the user from having to specify the time step.
13) The PΦ formulation is simple, easy for students to learn, and easy to apply.
14) Transformations are not necessary